CONFORMALLY OSSERMAN MANIFOLDS AND CONFORMALLY COMPLEX SPACE FORMS

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Abstract. We characterize manifolds which are locally conformally equivalent to either complex projective space or to its negative curvature dual in terms of their Weyl curvature tensor. As a byproduct of this investigation, we classify the conformally complex space forms if the dimension is at least 8. We also study when the Jacobi operator associated to the Weyl conformal curvature tensor of a Riemannian manifold has constant eigenvalues on the bundle of unit tangent vectors and classify such manifolds which are not conformally flat in dimensions congruent to 2 mod 4.

1. Introduction

1.1. **The Weyl curvature.** Let ∇ be the Levi-Civita connection of a Riemannian manifold (M,g) of dimension m. The curvature operator $R(x,y)$ and curvature tensor $R(x,y,z,w)$ are defined by setting:

$$R(x,y) := \nabla_x\nabla_y - \nabla_y\nabla_x - \nabla_{[x,y]} \quad\text{and}\quad R(x,y,z,w) = g(R(x,y)z,w)\,.$$

Let $\{e_i\}$ be a local orthonormal frame for the tangent bundle. We sum over repeated indices to define the *Ricci tensor* ρ and the *scalar curvature* τ by setting:

$$\rho_{ij} := \textstyle\sum_k R_{ikkj} \quad\text{and}\quad \tau := \sum_i \rho_{ii}\,.$$

The associated *Ricci operator* is defined by setting $\rho(e_i) = \sum_j \rho_{ij} e_j$. We introduce additional tensors by setting

$$\begin{aligned} L(x,y)z &:= g(\rho y,z)x - g(\rho x,z)y + g(y,z)\rho x - g(x,z)\rho y,\\ R_0(x,y)z &:= g(y,z)x - g(x,z)y\,. \end{aligned} \tag{1.a}$$

Let W be the *Weyl conformal curvature operator*. We may decompose

$$\begin{aligned} &R(x,y) = W(x,y) + c_1(m)\tau R_0(x,y) + c_2(m)L(x,y) \quad\text{where}\\ &c_1(m) := -\tfrac{1}{(m-1)(m-2)} \quad\text{and}\quad c_2(m) := \tfrac{1}{m-2}\,. \end{aligned} \tag{1.b}$$

1.2. **Conformal geometry.** We say that two Riemannian metrics g_1 and g_2 are *conformally equivalent* if $g_1 = \alpha\cdot g_2$ where α is a smooth positive scaling function. The Weyl conformal curvature operator is invariant on a conformal class as

$$W_{g_1} = W_{g_2} \quad\text{if}\quad g_1 = \alpha\cdot g_2\,. \tag{1.c}$$

Conformal analogues of notions in Riemannian geometry can be obtained by replacing the full curvature operator R by the Weyl operator W; we add the prefix "conformally" in doing this. For example, one says that (M,g) is *conformally flat* if the Weyl tensor W vanishes identically; this implies that (M,g) is conformally equivalent to flat space.

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1.3. **Space forms and complex space forms.** One says that (M,g) is a *space form* if $R=\lambda_0 R_0$ for some smooth function λ_0 or, equivalently, if (M,g) has pointwise constant sectional curvature. If $m\geq 3$, then necessarily λ_0 is constant and by rescaling the metric, we may assume $\lambda_0\in\{-1,0,1\}$. If $\lambda_0=-1$, then (M,g) is locally isometric to hyperbolic space; if $\lambda_0=0$, then (M,g) is locally isometric to flat space; if $\lambda_0=1$, then (M,g) is locally isometric to the sphere. Thus the geometry is very rigid in this setting.

Let Φ be a Hermitian almost complex structure on TM; necessarily $m=2n$ is even. We set

$$R_\Phi(x,y)z:=g(\Phi y,z)\Phi x-g(\Phi x,z)\Phi y-2g(\Phi x,y)\Phi z\,. \tag{1.d}$$

We say that (M,g) is a *complex space form* if $R=\lambda_0R_0+\lambda_1R_\Phi$ for smooth functions λ_0 and λ_1 where $\lambda_1\neq 0$. Let $(\mathbb{CP}^n,g_{FS})$ denote complex projective space with the Fubini-Study metric and let $({}^*\mathbb{CP}^n,{}^*g_{FS})$ be the negative curvature dual; these are complex space forms. Conversely, if (M,g) is a complex space form and if $m\geq 6$, then one can show that $\lambda_0=\lambda_1$ and that λ_0 is constant. By rescaling the metric, we may assume $\lambda_0=\pm1$. If $\lambda_0=1$, then (M,g) is locally isometric to $(\mathbb{CP}^n,g_{FS})$; if $\lambda_0=-1$, then (M,g) is locally isometric to $({}^*\mathbb{CP}^n,{}^*g_{FS})$. We refer to [12] for further details. A generalization of this result to the pseudo-Riemannian setting may be found in [8].

1.4. **Conformally complex space forms.** One says that (M,g) is a *conformal space form* if $W=\lambda_0R_0$; we shall see presently this implies $\lambda_0=0$ so (M,g) is conformally flat. Similarly one says that (M,g) is a *conformally complex space form* if $W=\lambda_0R_0+\lambda_1R_\Phi$ for some Hermitian almost complex structure on TM where λ_0 and λ_1 are smooth functions on M with $\lambda_1\neq0$. A similar rigidity result holds:

Theorem 1.1. *Let (M,g) be a conformally complex space form with $m\geq 8$. Then (M,g) is locally conformally equivalent to either $(\mathbb{CP}^n,g_{FS})$ or $({}^*\mathbb{CP}^n,{}^*g_{FS})$.*

We remark that our proof of Theorem 1.1 extends to the higher signature setting; we shall omit details in the interests of brevity.

1.5. **The Jacobi operator J_R.** We define a self-adjoint map $J_R(x)$ of the tangent bundle defined by setting:

$$J_R(x)y=R(y,x)x\,.$$

One says that (M,g) is *Osserman* if the eigenvalues of $J_R(x)$ are constant on the sphere bundle $S(M,g)$ of unit tangent vectors. One says that (M,g) is a *local* 2 *point homogeneous space* if the local isometries of (M,g) act transitively on $S(M,g)$; this necessarily implies that (M,g) is Osserman. Osserman [11] wondered if the converse held; this has been called the Osserman conjecture by subsequent authors. Chi [4] and Nikolayevsky [9, 10] established the Osserman conjecture if $m\neq 8,16$.

There are similar questions for pseudo-Riemannian manifolds. Let (M,g) be a pseudo-Riemannian manifold of signature (p,q). One says that (M,g) is spacelike (resp. timelike) Jordan Osserman if the Jordan normal form of J_R is constant on the pseudo-sphere of unit spacelike (resp. timelike) vectors. In the Lorentzian setting ($q=1$), it is known that any spacelike (resp. timelike) Jordan Osserman manifold has constant sectional curvature [2, 6]. The analogous question in the higher signature setting is far from settled. For example, there are spacelike and timelike Jordan Osserman pseudo-Riemannian manifolds which are not locally homogeneous and thus are not local symmetric spaces [7].

1.6. **The conformal Jacobi operator.** We follow the discussion of [3] and study a conformal analogue of the Osserman conjecture. The *conformal Jacobi operator* J_W is given by:

$$J_W(x)y := W(y,x)x;$$

the constants c_i are chosen so that

$$\mathrm{Tr}\{J_W\} = 0\,. \tag{1.e}$$

Thus, in particular, if $W = \lambda_0 R_0$, then $0 = \mathrm{Tr}(J_W(x)) = (m-1)\lambda_0 g(x,x)$ so $\lambda_0 = 0$ and (M,g) is conformally flat as noted above.

We say that (M,g) is *conformally Osserman* if the eigenvalues of J_W are constant on the fiber spheres

$$S_P(M,g) := \{x \in T_PM : g(x,x) = 1\};$$

the eigenvalues are allowed to vary from point to point. Since space forms and complex space forms are 2 point homogeneous spaces, they are examples of conformally Osserman manifolds. In particular, $(\mathbb{CP}^n, g_{FS})$ and $({}^*\mathbb{CP}^n, {}^*g_{FS})$ are conformally Osserman manifolds.

If $g_1 = \alpha g_2$, then $S_P(M,g_1) = \alpha^{-1/2} S_P(M,g_2)$. Furthermore $J_{W(g_1)} = J_{W(g_2)}$ by Equation (1.c). Thus the eigenvalues on the unit sphere bundles rescale so:

Theorem 1.2. *Let g_1 and g_2 be conformally equivalent metrics on M. Then (M,g_1) is conformally Osserman if and only if (M,g_2) is conformally Osserman.*

Since $J_W(x)x = 0$ and since $J_W(x)$ is self-adjoint, J_W preserves the subspace $x^\perp$. We define the *reduced conformal Jacobi operator* by letting

$$\tilde{J}_W(x) := J_W(x)|_{x^\perp}\,.$$

It is then immediate that J_W has constant eigenvalues on $S_P(M,g)$ if and only if $\tilde{J}_W$ has constant eigenvalues on $S_P(M,g)$; eliminating the trivial eigenvector x simplifies subsequent statements.

Suppose that (M,g) is locally conformally equivalent to a local 2 point homogeneous space (M_0,g_0). Since the local isometries of (M_0,g_0) act transitively on $S(M_0,g_0)$, the eigenvalues of J_{W_0} are constant on $S(M_0,g_0)$ so (M_0,g_0) is conformally Osserman. Thus by Theorem 1.2, (M,g) is conformally Osserman. The following two results are partial converses to this observation.

We can classify conformally Osserman manifolds with certain eigenvalue structures:

Theorem 1.3. *Let (M,g) be a conformally Osserman Riemannian manifold.*

(1) *Suppose that $\tilde{J}_{W_P}$ has only one eigenvalue at each point P of M. Then (M,g) is conformally flat.*
(2) *Let $m \geq 8$. Suppose that $\tilde{J}_{W_P}$ has two distinct eigenvalues of multiplicities 1 and $m-2$ for each point $P \in M$. Then (M,g) is locally conformally equivalent to either $(\mathbb{CP}^n, g_{FS})$ or $({}^*\mathbb{CP}^n, {}^*g_{FS})$.*

We can use topological methods to control the eigenvalue structure in certain dimensions and derive the following result from Theorem 1.3:

Theorem 1.4. *Let (M,g) be a conformally Osserman Riemannian manifold of dimension m.*

(1) *If m is odd, then (M,g) is conformally flat.*
(2) *If $m = 4k+2 \geq 10$ and if P is a point of M where $W_P \neq 0$, then there is an open neighborhood of P in M which is conformally equivalent to an open subset of either $(\mathbb{CP}^n, g_{FS})$ or $({}^*\mathbb{CP}^n, {}^*g_{FS})$.*

Here is a brief guide to the paper. In Section 2 we prove Theorem 1.1 and classify the conformally complex space forms. In Section 3, we review results of Chi [4] in the algebraic context. In Section 4, we establish Theorems 1.3 and 1.4 and thereby establish a conformal equivalent of the Osserman conjecture in certain situations.

2. Conformally complex space forms

We adopt an argument of Tricerri and Vanhecke [12] to establish Theorem 1.1. Let (M, g) be a conformally complex space form of dimension $m \geq 8$. By assumption, there exists a Hermitian almost complex structure Φ on M so that

$$W = \lambda_0 R_0 + \lambda_1 R_\Phi \quad \text{where} \quad \lambda_1 \neq 0\,.$$

If $g(x, x) = 1$, then:

$$J_W(x)y = \begin{cases} 0 & \text{if} \quad y = x. \\ (\lambda_0 + 3\lambda_1)y & \text{if} \quad y = \Phi x, \\ \lambda_0 y & \text{if} \quad y \perp \{x, \Phi x\}\,. \end{cases}$$

Thus $\tilde{J}_W$ has two eigenvalues $(\lambda_0 + 3\lambda_1, \lambda_0)$ with multiplicities $(1, m-2)$. Since $\operatorname{Tr}\{J_W\} = 0$, this shows

$$\begin{aligned} &3\lambda_1 + (m-1)\lambda_0 = 0 \quad \text{so} \quad \lambda_0 = -\tfrac{3}{m-1}\lambda_1 \quad \text{and} \\ &R = \lambda_1 R_\Phi + c_2(m)L + (c_1(m)\tau - \tfrac{3}{m-1}\lambda_1)R_0\,. \end{aligned} \tag{2.a}$$

If g_1 and g_2 are conformally related, then

$$W_{g_1} = W_{g_2}, \quad \Phi_{g_2} = \Phi_{g_1}, \quad \text{and} \quad \lambda_1(g_1)g_1 = \lambda_1(g_2)g_2\,.$$

Set $g_2 := |\lambda_1(g_1)|g_1$; $\lambda_1(g_2) = \pm 1$. Thus we may therefore assume henceforth without loss of generality that $\lambda_1 = \pm 1$. Let $\Phi_{;x}$, $W_{;x}$, and $R_{;x}$ denote the covariant derivatives of these tensors. Since any complex space form is locally isometric to complex projective space with a multiple of the Fubini-Study metric or to the negative curvature dual, Theorem 1.1 will follow the following result:

Lemma 2.1. *Let (M, g, Φ) satisfy $W = \lambda_1 R_\Phi + \lambda_0 R_0$ where $\lambda_1 = \pm 1$. Assume that $m \geq 8$. Let $\{a, \Phi a, b, \Phi b, c, \Phi c\}$ be an orthonormal set. Then:*

1. *We have $\Phi_{;a}\Phi = -\Phi\Phi_{;a}$. We also have Φ, $\Phi_{;a}$, and $\Phi\Phi_{;a}$ are skew-adjoint.*
2. *We have $(\Phi_{;c}b - \Phi_{;b}c, a) = 0$.*
3. *We have $\Phi_{;a}a = 0$, $\Phi_{;a}\Phi a = 0$, and $\Phi_{;a}b + \Phi_{;b}a = 0$.*
4. *We have $\nabla\Phi = 0$.*
5. *We have that (M, g, Φ) is a complex space form.*

Proof. We covariantly differentiate the identity $\Phi^2 = -\operatorname{Id}$ to see $\Phi_{;a}\Phi + \Phi\Phi_{;a} = 0$. As Φ is skew-adjoint, $\Phi_{;a}$ is skew-adjoint. The fact that $\Phi\Phi_{;a}$ is skew-adjoint then follows from the fact that $\Phi_{;a}$ and Φ anti-commute. Assertion (1) follows.

We use the second Bianchi identity

$$\{R_{;x}(y, z) + R_{;y}(z, x) + R_{;z}(x, y)\}w = 0 \tag{2.b}$$

to prove Assertions (2) and (3). Let $\sigma_{x,y,z}$ be summation with respect to the cyclic permutation of (x, y, z). Equations (2.a) and (2.b) imply:

$$\begin{aligned} 0 = \sigma_{x,y,z}\{&c_1\tau_{;x}[g(z, w)y - g(y, w)z] \\ &+c_2[g(\rho_{;x}z, w)y - g(\rho_{;x}y, w)z + g(z, w)\rho_{;x}y - g(y, w)\rho_{;x}z] \\ &+\lambda_1[g(\Phi_{;x}z, w)\Phi y + g(\Phi z, w)\Phi_{;x}y - g(\Phi_{;x}y, w)\Phi z - g(\Phi y, w)\Phi_{;x}z \\ &-2g(\Phi_{;x}y, z)\Phi w - 2g(\Phi y, z)\Phi_{;x}w]\}\,. \end{aligned} \tag{2.c}$$

Since $m \geq 8$, we may choose d so that $\{a, \Phi a, b, \Phi b, c, \Phi c, d, \Phi d\}$ is an orthonormal set. Let $x = a$, $y = b$, $z = c$, and $w = d$ in Equation (2.c). Then:

$$\begin{aligned} 0 &= \sigma_{a,b,c}\{c_2[g(\rho_{;a}c, d)b - g(\rho_{;a}b, d)c] \\ &+ \lambda_1[g(\Phi_{;a}c, d)\Phi b - g(\Phi_{;a}b, d)\Phi c - 2g(\Phi_{;a}b, c)\Phi d]\}\,. \end{aligned}$$

Since $\lambda_1 \neq 0$, $g(\Phi_{;a}c - \Phi_{;c}a, d)\Phi b = 0$. Assertion (2) follows by setting the coefficient of Φb to zero and permuting a, b, c and d appropriately.

To prove Assertion (3), set $x = a$, $y = b$, $z = \Phi b$, and $w = d$ in Equation (2.c):

$$\begin{aligned} 0 &= c_1\tau_{;a}[g(\Phi b, d)b - g(b, d)\Phi b] \\ &+ c_2[g(\rho_{;a}\Phi b, d)b - g(\rho_{;a}b, d)\Phi b + g(\Phi b, d)\rho_{;a}b - g(b, d)\rho_{;a}\Phi b] \\ &+ \lambda_1[g(\Phi_{;a}\Phi b, d)\Phi b + g(\Phi\Phi b, d)\Phi_{;a}b - g(\Phi_{;a}b, d)\Phi\Phi b - g(\Phi b, d)\Phi_{;a}\Phi b \\ &- 2g(\Phi_{;a}b, \Phi b)\Phi d - 2g(\Phi b, \Phi b)\Phi_{;a}d] \\ &+ c_1\tau_{;b}[g(a, d)\Phi b - g(\Phi b, d)a] \\ &+ c_2[g(\rho_{;b}a, d)\Phi b - g(\rho_{;b}\Phi b, d)a + g(a, d)\rho_{;b}\Phi b - g(\Phi b, d)\rho_{;b}a] \\ &+ \lambda_1[g(\Phi_{;b}a, d)\Phi\Phi b + g(\Phi a, d)\Phi_{;b}\Phi b - g(\Phi_{;b}\Phi b, d)\Phi a - g(\Phi\Phi b, d)\Phi_{;b}a \\ &- 2g(\Phi_{;b}\Phi b, a)\Phi d - 2g(\Phi\Phi b, a)\Phi_{;b}d] \\ &+ c_1\tau_{;\Phi b}[g(b, d)a - g(a, d)b] \\ &+ c_2[g(\rho_{;\Phi b}b, d)a - g(\rho_{;\Phi b}a, d)b + g(b, d)\rho_{;\Phi b}a - g(a, d)\rho_{;\Phi b}b] \\ &+ \lambda_1[g(\Phi_{;\Phi b}b, d)\Phi a + g(\Phi b, d)\Phi_{;\Phi b}a - g(\Phi_{;\Phi b}a, d)\Phi b - g(\Phi a, d)\Phi_{;\Phi b}b \\ &- 2g(\Phi_{;\Phi b}a, b)\Phi d - 2g(\Phi a, b)\Phi_{;\Phi b}d]\,. \end{aligned}$$

which simplifies to become:

$$\begin{aligned} 0 &= c_2[g(\rho_{;a}\Phi b, d)b - g(\rho_{;a}b, d)\Phi b] + c_2[g(\rho_{;b}a, d)\Phi b - g(\rho_{;b}\Phi b, d)a] \\ &+ c_2[g(\rho_{;\Phi b}b, d)a - g(\rho_{;\Phi b}a, d)b] \\ &+ \lambda_1[g(\Phi_{;a}\Phi b, d)\Phi b + g(\Phi_{;a}b, d)b - 2g(\Phi_{;a}b, \Phi b)\Phi d - 2\Phi_{;a}d]\} \\ &+ \lambda_1[-g(\Phi_{;b}a, d)b - g(\Phi_{;b}\Phi b, d)\Phi a - 2g(\Phi_{;b}\Phi b, a)\Phi d] \\ &+ \lambda_1[(g(\Phi_{;\Phi b}b, d)\Phi a - g(\Phi_{;\Phi b}a, d)\Phi b - 2g(\Phi_{;\Phi b}a, b)\Phi d]\,. \end{aligned} \tag{2.d}$$

We apply Assertions (1) and (2) to see

$$\begin{aligned} 0 &= \{g(\rho_{;a}\Phi b, d) - g(\rho_{;\Phi b}a, d)\}b = \{-g(\rho_{;a}b, d) + g(\rho_{;b}a, d)\}\Phi b, \\ 0 &= \{g(\Phi_{;a}\Phi b, d) - g(\Phi_{;\Phi b}a, d)\}\Phi b = \{g(\Phi_{;a}b, d) - g(\Phi_{;b}a, d)\}b \\ 0 &= g(\Phi_{;a}b, \Phi b), \quad \text{and} \quad g(\Phi_{;\Phi b}a, b) = -g(\Phi_{;\Phi b}b, a)\,. \end{aligned}$$

Consequently, we may rewrite Equation (2.d) in the form:

$$\begin{aligned} 0 &= c_2[-g(\rho_{;b}\Phi b, d)a + g(\rho_{;\Phi b}b, d)a] - 2\lambda_1\Phi_{;a}d \\ &- 2\lambda_1\{g(\Phi_{;b}\Phi b, a) - g(\Phi_{;\Phi b}b, a)\}\Phi d \\ &+ \lambda_1\{-g(\Phi_{;b}\Phi b, d) + g(\Phi_{;\Phi b}b, d)\}\Phi a\,. \end{aligned}$$

As by Assertion (1) $\Phi\Phi_{;a}$ is skew-adjoint, $g(\Phi_{;a}d, \Phi d) = 0$. Taking the inner products with Φd and with Φa then yields

$$0 = g(\Phi_{;b}\Phi b - \Phi_{;\Phi b}b, a), \tag{2.e}$$

$$0 = g(\Phi_{;\Phi b}b - \Phi_{;b}\Phi b, d) - 2\lambda_1 g(\Phi_{;a}d, \Phi a)\,. \tag{2.f}$$

Equation (2.e) shows that $(\Phi_{;b}\Phi b - \Phi_{;\Phi b}b) \in \operatorname{Span}\{b, \Phi b\}$. Since

$$g(\Phi_{\Phi b}b, b) = g(\Phi_{\Phi b}b, \Phi b) = g(\Phi_{;b}\Phi b, b) = g(\Phi_{;b}\Phi b, \Phi b) = 0, \tag{2.g}$$

we may conclude $\Phi_{\Phi b}b - \Phi_{;b}\Phi b = 0$. Equation (2.f) then implies

$$0 = -2g(\Phi_{;a}d, \Phi a) = 2g(d, \Phi_{;a}\Phi a)\,.$$

This implies $\Phi_{;a}\Phi a \in \operatorname{Span}\{a, \Phi a\}$ and applying Equation (2.g) shows $\Phi_{;a}\Phi a = 0$. We then see $\Phi_{;a}a = -\Phi\Phi\Phi_{;a}a = \Phi\Phi_{;a}\Phi a = \Phi 0 = 0$. The final identity of Assertion (3) then follows by polarization.

By Assertions (2) and (3), $g(\Phi_{;a}b, c) = 0$ so $\Phi_{;a}b \in \operatorname{Span}\{a, \Phi a, b, \Phi b\}$. We show $\Phi_{;a}b = 0$ and establish Assertion (4) by computing:

$$\begin{aligned}
0 &= g(\Phi_{;a}b, a) = -g(\Phi_{;a}a, b) = 0,\\
0 &= g(\Phi_{;a}b, \Phi a) = -g(\Phi_{;a}\Phi a, b) = 0,\\
0 &= g(\Phi_{;a}b, b) = g(\Phi_{;a}b, \Phi b)\,.
\end{aligned}$$

Because $\nabla\Phi = 0$, $R(x,y)\Phi = \Phi R(x,y)$. We compute:

$$\begin{aligned}
&R(x,y)\Phi z = \lambda_1\{g(y,z)\Phi x - g(x,z)\Phi y + 2g(\Phi x, y)z\}\\
&\quad +c_2(m)\{g(\rho y, \Phi z)x - g(\rho x, \Phi z)y + g(y, \Phi z)\rho x - g(x, \Phi z)\rho y\}\\
&\quad +(c_1(m)\tau - \tfrac{3}{m-1}\lambda_1)\{g(y, \Phi z)x - g(x, \Phi z)y\}\\
&\Phi R(x,y)z = \lambda_1\{-g(\Phi y, z)x + g(\Phi x, z)y + 2g(\Phi x, y)z\}\\
&\quad +c_2(m)\{g(\rho y, z)\Phi x - g(\rho x, z)\Phi y + g(y,z)\Phi\rho x - g(x,z)\Phi\rho y\}\\
&\quad +(c_1(m)\tau - \tfrac{3}{m-1}\lambda_1)\{g(y,z)\Phi x - g(x,z)\Phi y\}\,.
\end{aligned} \tag{2.h}$$

Let $x = a$, $y = b$, and $z = a$ in Equation (2.h):

$$\begin{aligned}
&\lambda_1\{-\Phi b\} + c_2(m)\{g(\rho b, \Phi a)a - g(\rho a, \Phi a)b\}\\
= \;& c_2(m)\{g(\rho b, a)\Phi a - g(\rho a, a)\Phi b - \Phi\rho b\} + (c_1(m)\tau - \tfrac{3}{m-1}\lambda_1)\{-\Phi b\}\,.
\end{aligned} \tag{2.i}$$

This implies $\Phi\rho b \in \operatorname{Span}\{\Phi b, \Phi a, b, a\}$. Similarly $\Phi\rho b \in \operatorname{Span}\{\Phi b, \Phi c, b, c\}$. This implies $\rho b \in \operatorname{Span}\{b, \Phi b\}$. Taking the inner product of Equation (2.i) with b we get $-g(\rho a, \Phi a) = -g(\Phi\rho b, b) = g(\rho b, \Phi b)$. Symmetrizing over $\{a, b, c\}$, we see $g(\rho b, \Phi b) = 0$ so $\rho b \in \operatorname{Span}\{b\}$ and hence $\rho b = \mu(b)b$. As

$$\begin{aligned}
&\rho b = \mu(b)b, \quad \rho a = \mu(a)a, \quad \text{and,}\\
&\rho(a+b) = \mu(a+b)(a+b) = \mu(a)a + \mu(b)b,
\end{aligned}$$

$\mu(a+b) = \mu(a) = \mu(b)$ and hence μ is constant. This implies (M, g) is Einstein. Consequently, L_0 is a multiple of R_0. Equation (2.a) now implies that (M, g, Φ) is a complex space form. □

3. Algebraic curvature tensors

We consider a triple $\mathcal{V} := (V, g, A)$ where g is a positive definite inner product on a real vector space V of dimension m and where $A \in \otimes^4 V^*$ is an *algebraic curvature tensor* on V; i.e. A has the usual symmetries of the Riemann curvature tensor:

$$\begin{aligned}
&A(x,y,z,w) = A(z,w,x,y) = -A(y,x,z,w), \quad \text{and}\\
&A(x,y,z,w) + A(y,z,x,w) + A(z,x,y,w) = 0\,.
\end{aligned}$$

We follow the discussion in [8]. If Ψ is a self-adjoint map of V, set:

$$A_\Psi(x,y,z,w) := g(\Psi x, w)g(\Psi y, z) - g(\Psi x, z)g(\Psi y, w)\,. \tag{3.a}$$

For example, the algebraic curvature tensors L and R_0 of Equation (1.a) can be expressed in the form:

$$R_0 = R_{\mathrm{Id}} \quad \text{and} \quad L = R_{\mathrm{Id}+\rho} - R_{\mathrm{Id}} - R_\rho\,.$$

Similarly if Φ is skew-adjoint, we generalize Equation (1.d) and set:

$$A_\Phi(x,y,z,w) := g(\Phi x, w)g(\Phi y, z) - g(\Phi x, z)g(\Phi y, w) - 2g(\Phi x, y)g(\Phi z, w)\,. \tag{3.b}$$

One checks easily that the tensors A_Ψ and A_Φ defined above are algebraic curvature tensors. One has the following result of Fiedler [5]:

Theorem 3.1. *The space of all algebraic curvature tensors is a real vector space. It is spanned by the tensors A_Ψ of Equation (3.a). It is also spanned by the tensors A_Φ of Equation (3.b).*

If A is an algebraic curvature tensor, then we define the associated curvature operator $A(x,y)$ and Jacobi operator $J_A(x)$ by the relations:

$$g(A(x,y)z,w) = A(x,y,z,w) \quad \text{and} \quad g(J_A(x)y,z) = A(y,x,x,z)\,.$$

The operator $J_A(x) = A(\cdot,x)x$ is a self-adjoint map of V and we say that $\mathcal{V}$ is *Osserman* if the eigenvalues of J_A are constant on $S(V) := \{x \in V : g(x,x) = 1\}$. We note that $J_A(x)x = 0$ and let $\tilde{J}_A(x)$ be the restriction of $J_A(x)$ to $x^\perp$.

The following classification result is due to Chi [4] and will be crucial in establishing Theorem 1.3.

Theorem 3.2. *Let A be an Osserman algebraic curvature tensor on $\mathcal{V}$.*
1. *If $\tilde{J}_A$ has only one eigenvalue, then $A = \lambda A_{\mathrm{Id}}$.*
2. *If $\tilde{J}_A$ has two eigenvalues and if one of those eigenvalues has multiplicity 1, then there exists a Hermitian almost complex structure Φ on V and there exist real constants λ_0 and λ_1 so that $A = \lambda_0 A_{\mathrm{Id}} + \lambda_1 A_\Phi$.*

The following observation is also due to Chi [4]; it is a straightforward application of work of Adams [1] concerning vector fields on spheres and will be critical in proving Theorem 1.4:

Theorem 3.3. *Let A be an Osserman algebraic curvature tensor on $\mathcal{V}$.*
1. *If m is odd, then $\tilde{J}_A$ has only one eigenvalue.*
2. *If $m \equiv 2$ mod 4, then either $\tilde{J}_A$ has only eigenvalue or $\tilde{J}_A$ has exactly 2 eigenvalues and one of those eigenvalues has multiplicity 1.*

4. Conformal Osserman manifolds

Proof of Theorem 1.3 (1). Assume that (M,g) is conformally Osserman and that $\tilde{J}_W$ has only one eigenvalue. By Theorem 3.2, $W_P = \lambda_P A_{\mathrm{Id}}$. By Equation (1.e),

$$0 = \mathrm{Tr}\{J_W(x)\} = (m-1)\lambda_P g(x,x)\,.$$

Thus $\lambda_P = 0$ so $W = 0$ and (M,g) is conformally flat. □

Proof of Theorem 1.3 (2). Assume that (M,g) is conformally Osserman, that $\tilde{J}_W$ has two distinct eigenvalues, that one of the eigenvalues has multiplicity 1, and that $m \geq 8$. By Theorem 3.2, there exists a Hermitian almost complex structure Φ on each tangent space so that

$$W = \lambda_0 R_{\mathrm{Id}} + \lambda_1 R_\Phi \quad \text{where} \quad \lambda_0 = -\tfrac{3}{m-1}\lambda_1\,.$$

We then use techniques developed in [8] to show that Φ can be chosen to vary smoothly with P, at least locally. Thus (M,g) is a conformally complex space form so Theorem 1.3 (2) follows from Theorem 1.1. □

Proof of Theorem 1.4 (1). Let (M,g) be conformally Osserman. If m is odd, then Theorem 3.3 implies that $\tilde{J}_W$ has only one eigenvalue. Therefore by Theorem 1.3, (M,g) is conformally flat. □

Proof of Theorem 1.4 (2). Let $m = 4k+2 \geq 10$ and let P be a point of M where $W_P \neq 0$. By Theorem 3.3, either $\tilde{J}_{W_P}$ has only one eigenvalue or $\tilde{J}_{W_P}$ has two eigenvalues and one has multiplicity 1. If $\tilde{J}_{W_P}$ has only one eigenvalue, then that eigenvalue is 0 by Equation (1.e). This implies $J_{W_P} = 0$ and hence $W_P = 0$ which is contrary to the assumption which we have made. Consequently $\tilde{J}_{W_P}$ has two eigenvalues at P and hence on a neighborhood $\mathcal{O}_P$. By Theorem 1.3 (2), $(\mathcal{O}_P, g)$ is

locally conformally equivalent to an open subset of either complex projective space with the Fubini study metric or to the negative curvature dual. □

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References

[1] J. Adams, *Vector fields on spheres*, Annals of Math. **75** (1962), 603–632.

[2] N. Blažić, N. Bokan and P. Gilkey, *A Note on Osserman Lorentzian manifolds*, Bull. London Math. Soc. **29** (1997), 227–230.

[3] N. Blažić, P. Gilkey, S. Nikčević, and U. Simon, *The spectral geometry of the Weyl conformal tensor*, math.DG/0310226.

[4] Q.-S. Chi, *A curvature characterization of certain locally rank-one symmetric spaces*, J. Differential Geom. **28** (1988), 187–202.

[5] B. Fiedler, *Determination of the structure of algebraic curvature tensors by means of Young symmetrizers*, Seminaire Lotharingien de Combinatoire B48d (2003). 20 pp. Electronically published: http://www.mat.univie.ac.at/∼slc/; see also math.CO/0212278.

[6] E. García–Río, D. Kupeli and M. E. Vázquez-Abal, *On a problem of Osserman in Lorentzian geometry*, Differential Geom. Appl. **7** (1997), 85–100.

[7] E. García–Río, D. Kupeli, and R. Vázquez-Lorenzo, **Osserman Manifolds in Semi-Riemannian Geometry**, Lecture Notes in Mathematics, 1777. Springer-Verlag, Berlin, 2002.

[8] P. Gilkey, **Geometric properties of natural operators defined by the Riemann curvature tensor**, World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[9] Y. Nikolayevsky, *Two theorems on Osserman manifolds*, Differential Geom. Appl. **18** (2003), 239–253.

[10] Y. Nikolayevsky, *Osserman Conjecture in dimension* $n \neq 8, 16$; math.DG/0204258.

[11] R. Osserman, *Curvature in the eighties*, Amer. Math. Monthly, **97**, (1990) 731–756.

[12] F. Tricerri and L. Vanhecke, *Curvature tensors on almost Hermitian manifolds*, Trans. Amer. Math. Soc. **267** (1981), 365–397.

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